

# Boson-Fermion Resonance Model in One Dimension

A. Recati<sup>1,2</sup>, J.N. Fuchs<sup>1,3</sup>, and W. Zwerger<sup>1</sup>

<sup>1</sup>*Institute for Theoretical Physics, Universität Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria*

<sup>2</sup>*CRS BEC-INFM, Povo and ECT\*, Villazzano, I-38050 Trento, Italy*

<sup>3</sup>*Laboratoire de Physique des Solides, Université Paris-Sud, Bâtiment 510, F-91405 Orsay, France*

(Dated: February 2, 2008)

We discuss the BCS-BEC crossover for one-dimensional spin 1/2 fermions at zero temperature using the Boson-Fermion resonance model in one dimension. We show that in the limit of a broad resonance, this model is equivalent to an exactly solvable single channel model, the so-called modified Gaudin-Yang model. We argue that the one-dimensional crossover may be realized either via the combination of a Feshbach resonance and a confinement induced resonance or using direct photo-association in a two-component Fermi gas with effectively one-dimensional dynamics. In both cases, the system may be driven from a BCS-like state through a molecular Tonks-Girardeau gas close to resonance to a weakly interacting Bose gas of dimers.

PACS numbers: 03.75.Ss, 03.75.Hh, 74.20.Fg

## I. INTRODUCTION

In this article, we consider the problem of attractive fermions in one dimension (1D), having in mind current experiments on ultra-cold two-component Fermi gases of atoms [1, 2]. In these systems, the  $s$ -wave interaction between fermions in different internal states can be tuned using a Feshbach resonance. By changing the interaction from weakly attractive to weakly repulsive via a resonance where the interaction diverges, one can explore the crossover from a BCS superfluid, when the attraction is weak and pairing only appears in momentum space, to a Bose-Einstein condensate (BEC) of molecular dimers [3]. Experiments are currently investigating gases that are in a three dimensional regime (3D). A different situation occurs if the gas is confined in a very anisotropic cigar-shaped trap, like, e.g., in an atomic wire created with optical lattices [4] or on an atom chip [5]. If the transverse confinement is strong enough, the system effectively becomes 1D, i.e., the radial degrees of freedom are frozen. We will refer to such a situation as quasi-1D. In this case, at zero temperature, a crossover takes place between a BCS-like state and a weakly interacting Bose gas of dimers. This crossover can be described by an exactly solvable model (the so-called modified Gaudin-Yang model) [6, 7], which is just a combination of the Gaudin-Yang model for attractive fermions [8] and of the Lieb-Liniger model for repulsive dimers [9]. Despite the fact, that there is no genuine off-diagonal long range order in 1D even at zero temperature, we refer to this situation as a one dimensional version of the BCS-BEC crossover.

Such a crossover can be realized in two rather different ways using a two-component Fermi gas in a quasi-1D situation. They correspond to

- i) fermions whose 3D-scattering length exhibits a Feshbach resonance (FBR). In this case the combination of the 3D FBR and the confinement in the transverse direction, characterized by a trapping frequency  $\omega_{\perp}/2\pi$ , leads to a confinement induced

(CI) resonance [10], beyond which the two particle bound state energy is large enough to neglect breaking of dimers (this scenario has been discussed in [6, 7]).

- ii) Fermions which are transferred directly into a bound molecular state by an external laser field. The photo-association process can be described by an effective 1D Boson-Fermion resonance model (BFRM) [11, 12, 13]. For positive detuning of the laser this describes a system of attractively interacting fermions while for negative detuning, one obtains again unbreakable dimers for strong enough laser coupling.

The purpose of the present paper is to study the 1D BFRM, ii), at zero temperature. It will be shown that the resonance, which is reached by quite different means in both cases, quite generally allows driving a BCS-BEC crossover in 1D. In particular we will find that in the BFRM the molecular size on resonance  $r_{\star}$  plays a role similar to that of the transverse oscillator length  $a_{\perp} \equiv \sqrt{\hbar/m\omega_{\perp}}$  in the quasi-1D single channel model, i). In the limit of low density  $n$ , characterized either by  $na_{\perp} \ll 1$  or by  $nr_{\star} \ll 1$  respectively, the resonance is broad and both models are completely equivalent to the exactly solvable modified Gaudin-Yang model discussed in ref. [6, 7].

The paper is organized as follows: in Sec. II we introduce the model and the notations; Sec. III discusses the two-body problem, i.e., bound state and scattering properties; the many-body problem is addressed in Sec. IV using a functional integral approach; and in Sec. V we discuss the results.

## II. BOSON-FERMION RESONANCE MODEL

The Boson-Fermion resonance model [11, 13] is characterized by the following (grand-canonical) Hamiltonian

operator

$$\begin{aligned}\hat{H}' &= \hat{H} - \mu\hat{N} = \int dx \left( \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}_\sigma^\dagger \left[ -\frac{\hbar^2}{2m} \partial_x^2 - \mu \right] \hat{\psi}_\sigma \right. \\ &\quad \left. + \hat{\psi}_B^\dagger \left[ -\frac{\hbar^2}{4m} \partial_x^2 - 2\mu + \nu \right] \hat{\psi}_B + g \left( \hat{\psi}_B^\dagger \hat{\psi}_\uparrow \hat{\psi}_\downarrow + h.c. \right) \right) \quad (1)\end{aligned}$$

where  $\hat{\psi}_\sigma(x)$  (resp.  $\hat{\psi}_B(x)$ ) are fermionic (resp. bosonic) field operators describing atoms (resp. the bound state in the closed channel, i.e., bare dimers),  $\sigma$  identifies the spin projection  $\uparrow$  or  $\downarrow$ , corresponding to the two components in the Fermi gas,  $\mu$  is a Lagrange multiplier to be later identified with the chemical potential,  $m$  (resp.  $2m$ ) is the mass of the atoms (resp. of the bare dimers),  $\nu$  is the detuning in energy of one bare dimer with respect to two atoms and  $g$  is the coupling constant for the conversion of two atoms into a bare dimer and vice-versa. The model Eq. (1) can describe photo-association of molecules in a 1D geometry. In this context, the coupling constant  $g$  is determined by the matrix element of the dipole energy, i.e., the effective Rabi frequency, and a Franck-Condon-factor, arising from the overlap of the wave functions of atoms and molecules. We assume the molecular size to be much smaller than the oscillator length. Moreover we neglect the background scattering between fermions, i.e., we do not include terms of the form  $g_1^{bg} \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \hat{\psi}_\uparrow \hat{\psi}_\downarrow$  in the Hamiltonian. This is justified in any case close enough to resonance, i.e., where  $\nu \sim 0$  (see also Eq. (8)).

The operator measuring the total number of atoms (i.e. unbound atoms and atoms bound into bare dimers) is:

$$\hat{N} = \int dx \left( \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}_\sigma^\dagger \hat{\psi}_\sigma + 2\hat{\psi}_B^\dagger \hat{\psi}_B \right). \quad (2)$$

We consider the zero temperature behavior of a system made of  $N/2$  atoms with spin  $\uparrow$  and  $N/2$  atoms with spin  $\downarrow$  confined on a ring of length  $L$  and use the parameter  $\mu$  to insure that  $\langle \hat{N} \rangle = N$ . The thermodynamic limit is taken by letting  $N \rightarrow +\infty$  while maintaining the density  $n \equiv N/L$  fixed. From now on, we set  $\hbar = 1$ .

We note that the form of the local conversion term in Hamiltonian (1) is fixed by the Pauli principle in a two-component Fermi system. By contrast, for bosonic atoms, infinitely many local conversion terms  $g_l(\hat{\psi}^\dagger)^l \hat{\psi}_B^{(l)}$  with  $l = 2, 3, 4, \dots$  are possible and have to be considered. In order to understand the relevance of such terms, including also a possible background interaction between atoms, we performed a perturbative renormalization group (RG) analysis of the bosonized version of this model, i.e., an atomic Bose Luttinger liquid converting into molecular Bose Luttinger liquids. We find that if the background interaction is weak, essentially all conversion terms are relevant. This implies that it is impossible to describe 1D bosonic atoms close to resonance in terms of only a few parameters.

### III. TWO-BODY PROBLEM: SCATTERING AMPLITUDE AND BOUND STATE

In this section, we compute the molecular propagator in presence of only two atoms ( $N = 2$ ). From it, we obtain the scattering amplitude between two atoms and the dressed (i.e. renormalized) rest energy of a dimer. The latter corresponds to the energy of a two-atoms bound state. In momentum-energy space, the molecular propagator is given by

$$D(k, \omega) = D_0(k, \omega) + D_0(k, \omega) \Pi(k, \omega) D(k, \omega) \quad (3)$$

where  $D_0$  is the bare molecular propagator

$$D_0(k, \omega) = \left[ \omega - \frac{k^2}{4m} + 2\mu - \nu + i0^+ \right]^{-1} \quad (4)$$

and the “polarization”, i.e., self-energy of the closed channel propagator,  $\Pi(k, \omega)$  is given by:

$$\Pi(k, \omega) = g^2 \int \frac{dk'}{2\pi} \left[ \omega - k'^2/m - k^2/4m + 2\mu + i0^+ \right]^{-1}. \quad (5)$$

From Eq. (3), we can compute the dressed rest energy  $\epsilon_b$  of a dimer, which is defined as being the  $k = 0$  pole of the molecular propagator when  $\mu = 0$ :

$$D(0, \epsilon_b)^{-1} = D_0(0, \epsilon_b)^{-1} - \Pi(0, \epsilon_b) = 0. \quad (6)$$

We find that Eq. (6) admits a unique real negative solution  $|\epsilon_b| = -\epsilon_b$  irrespective of the sign of the detuning  $\nu$

$$\frac{|\epsilon_b|}{|\epsilon_\star|} - \sqrt{\frac{|\epsilon_\star|}{|\epsilon_b|}} + \frac{\nu}{|\epsilon_\star|} = 0, \quad (7)$$

where we have introduced the on-resonance ( $\nu = 0$ ) bound state energy  $\epsilon_\star \equiv \epsilon_b(\nu = 0) = -m^{1/3}g^{4/3}/2^{2/3}$ . This has to be compared with the 3D BFRM where a bound state is present only when the detuning is negative [14].

According to standard scattering theory, the  $T$ -matrix is given by  $T = g^2 D$  (see, e.g., [15]). Therefore, in the BFRM, the Lippmann-Schwinger equation for atoms is equivalent to the closed channel Dyson equation for the molecular propagator in vacuum, equation (3). From the latter it is possible to show that the scattering between two atoms can be described as resulting from an effective contact potential  $g_1 \delta(x)$ , which is a well defined 1D potential, with a bare scattering amplitude

$$g_1 \equiv g^2 D_0(0, 0) = -\frac{g^2}{\nu}. \quad (8)$$

When the detuning goes to zero, the bare scattering amplitude diverges: this corresponds to the resonance. Before resonance, we have  $\nu > 0$  and an attractive effective interaction  $g_1 < 0$  between the atoms, while after resonance  $\nu < 0$  and the effective interaction is repulsive

$g_1 > 0$ . Solving the Lippmann-Schwinger equation for the on-shell  $T$ -matrix of the contact potential

$$T(k', k, \Omega) = g_1 + i \int \frac{dk''}{2\pi} \frac{d\omega''}{2\pi} g_1 T(k', k'', \Omega) \times \left[ (\Omega/2 + \omega'' - k''^2/2m + i0^+) \times (\Omega/2 - \omega'' - k''^2/2m + i0^+) \right]^{-1} \quad (9)$$

in the limit  $k' = k \rightarrow 0$  and  $\Omega = k^2/m$ , we obtain the low-energy limit of the one-dimensional two-body  $T$ -matrix:

$$T_k = g^2 D(0, k^2/m) \simeq \frac{g_1}{1 + img_1/2k}. \quad (10)$$

The associated dressed scattering amplitude

$$f(k) = \frac{m}{2ik} T_k \simeq -\frac{1}{1 + ika_1} \quad (11)$$

has the standard form for 1D low energy scattering with  $a_1 \equiv -2/mg_1$  the 1D scattering length. It is a well-known fact that the 1D delta-potential forbids transmission at low scattering energy, i.e.,  $f(k) \rightarrow -1$  in the  $k \rightarrow 0$  limit [10].

Before studying the many-body problem, we would like to discuss briefly the behavior of the bound state energy. We define the size of the bound state as  $r_b \equiv (m|\epsilon_b|/2)^{-1/2}$ , which is finite for any detuning, and call  $r_\star \equiv r_b(\nu = 0)$  the size of the bound state on resonance. We find it useful also to define a dimensionless detuning  $\delta \equiv \nu/\epsilon_\star \sqrt{2}$ . With these definitions, equation (8) becomes  $g_1 = 2/mr_\star \delta$ .

In the BCS limit (i.e. when  $\delta \rightarrow -\infty$ ), the bound state energy can be written as

$$\epsilon_b \simeq -mg^2/4\nu^2 = -mg_1^2/4 \quad (12)$$

which agrees with the bound state energy of the  $g_1\delta(x)$  potential when  $g_1 < 0$ . In the opposite limit (BEC limit, i.e. when  $\delta \rightarrow +\infty$ ), the bound state energy is equal to the detuning

$$\epsilon_b \simeq \nu \quad (13)$$

and thus completely independent of the coupling constant  $g$ . The bound state energy  $\epsilon_b$  is plotted as a function of the dimensionless detuning  $\delta$  in Figure 1.

The behavior of the bound state in the 1D BFRM is qualitatively similar to that of the confinement induced bound state found by Bergeman, Moore and Olshanii [16] for two atoms trapped in a quasi-1D geometry (i.e., a waveguide with radial frequency  $\omega_\perp/2\pi$ ). This fact reveals the connection, at the two-body level, between the 1D BFRM and the quasi-1D single channel model. In Figure 2 we have plotted the confinement induced (CI) bound state as a function of  $\delta'$  (see below). In the quasi-1D case, the role of the dimensionless detuning  $\delta$  is played by the parameter  $\delta' \equiv a_\perp/a - A$ , where  $a_\perp \equiv (m\omega_\perp)^{-1/2}$  is the radial oscillator length,  $a$  is the

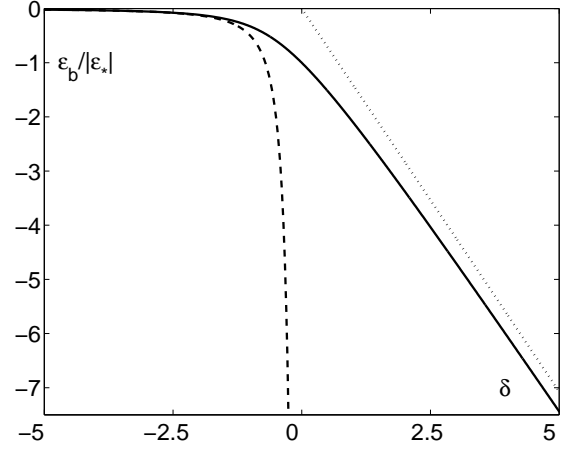


FIG. 1: Two-body bound state energy  $\epsilon_b$  [in units of the bound state energy on resonance  $|\epsilon_\star|$ ] as a function of the dimensionless detuning  $\delta$  (full line). The dashed line corresponds to the asymptotic behavior  $\epsilon_b \simeq -mg_1^2/4$  and the dotted line to the asymptotic behavior  $\epsilon_b \simeq \nu$ .

3D scattering length and  $A \equiv -\zeta(1/2, 1)/\sqrt{2} \simeq 1.0326$  [17]. In the quasi-1D geometry, the 1D scattering amplitude shows a CI resonance [10] and is given by

$$g'_1 \equiv 2\omega_\perp a(1 - Aa/a_\perp)^{-1} = 2/ma_\perp \delta' \quad (14)$$

which is similar to  $g_1 = 2/mr_\star \delta$ , showing that  $a_\perp$  plays the role of  $r_\star$ . The CI bound state energy  $\epsilon'_b$  obeys the following equation

$$\sqrt{2}a_\perp/a + \zeta(1/2, \epsilon'_b/\epsilon'_\star) = 0 \quad (15)$$

where  $\zeta(1/2, x)$  is a particular Hurwitz zeta function [16] and  $\epsilon'_\star \equiv \epsilon'_b(\delta' = 0) = -2/ma_\perp^2$  is the CI bound state energy on resonance. When  $\delta' \rightarrow -\infty$ ,  $\epsilon'_b \simeq -mg_1^2/4$ , in complete analogy with  $\epsilon_b \simeq -mg_1^2/4$ . On resonance  $\delta' = 0$ ,  $\epsilon'_\star = -2/ma_\perp^2$ , to be compared with  $\epsilon_\star = -2/mr_\star^2$ . After resonance when  $\delta' \rightarrow +\infty$ , the CI bound state energy behaves as  $\epsilon'_b \simeq -1/ma^2$ , which translates into  $\epsilon'_b/\epsilon'_\star \simeq (\delta' + A)^2/2 \simeq \delta'^2/2$  in terms of the parameter  $\delta'$  introduced above. Similarly, in the BFRM, the bound state energy decreases monotonically with the behavior  $\epsilon_b/\epsilon_\star \simeq \sqrt{2}\delta$  as a function of the dimensionless detuning  $\delta$ .

#### IV. MANY-BODY PROBLEM

The grand partition function at temperature  $T \equiv 1/\beta$  and chemical potential  $\mu$  can be written as a path integral

$$Z = \int \mathcal{D}(\bar{\psi}_\sigma, \psi_\sigma) \mathcal{D}(\bar{\psi}_B, \psi_B) e^{-S} \quad (16)$$

over Grassmann fields  $\bar{\psi}_\sigma, \psi_\sigma$  with  $\sigma = \uparrow, \downarrow$  and complex fields  $\bar{\psi}_B, \psi_B$  [18]. The action corresponding to the

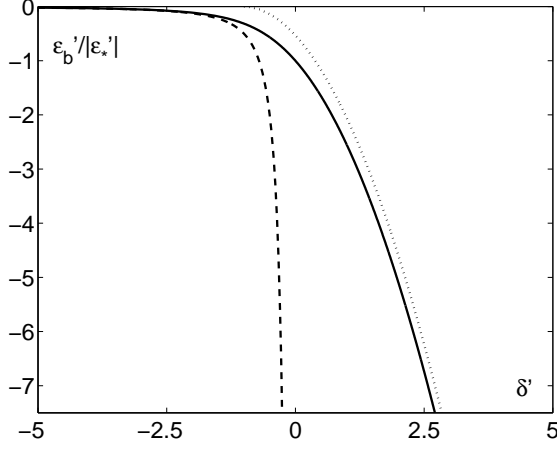


FIG. 2: Confinement induced bound state energy  $\epsilon'_b$  [in units of the bound state energy on resonance  $|\epsilon_\star|$ ] as a function of the dimensionless parameter  $\delta'$  (full line). The dashed line corresponds to the asymptotic behavior  $\epsilon'_b \simeq -mg_1^2/4$  and the dotted line to the asymptotic behavior  $\epsilon'_b \simeq -1/ma^2$ .

Hamiltonian (1) is:

$$S = \int_0^\beta d\tau \int dx \left( \sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \left[ \partial_\tau - \frac{\partial_x^2}{2m} - \mu \right] \psi_\sigma + \bar{\psi}_B \left[ \partial_\tau - \frac{\partial_x^2}{4m} - 2\mu + \nu \right] \psi_B + g \left( \bar{\psi}_B \psi_\uparrow \psi_\downarrow + c.c. \right) \right). \quad (17)$$

The average total number of atoms is obtained from

$$\langle N \rangle = \frac{\partial F}{\partial \mu} \quad (18)$$

where  $F \equiv -T \ln Z$  is the grand potential and we are interested in the  $T \rightarrow 0$  limit.

Let us define the Fermi momentum  $k_F \equiv \pi n/2$  and the Fermi energy  $\epsilon_F \equiv k_F^2/2m$  for an ideal gas of  $N$  spin 1/2 fermions. We shall study the particular case where the (modulus of the) energy of the two body bound state on resonance is much larger than the Fermi energy. This corresponds to the limit of a broad resonance (or strong coupling limit):

$$|\epsilon_\star| \gg \epsilon_F \Leftrightarrow nr_\star \ll 1 \Leftrightarrow g\sqrt{n} \gg \epsilon_F \quad (19)$$

The above inequality shows that the broad resonance limit corresponds to having a deeply bound state after resonance, i.e., that the dimers are unbreakable after resonance. We will show that in this limit, the system is described by a single channel model of atoms (fermions) only, before resonance, and of dimers (bosons) only, after resonance. In other words, the 1D BFRM in the broad resonance limit is equivalent to the modified Gaudin-Yang model. We mention that equation (19) is similar to the usual criterion for a broad 3D Feshbach resonance [19, 20].

### A. Before resonance: integrating out the bare dimers

Before resonance  $\nu > 0$ , it is possible to integrate out the bosonic fields and describe the system in terms of an effective action for fermions only. In order to show this, we need to define the Fourier transform of a field  $\psi(x, \tau)$ :

$$\psi(k, \tilde{\omega}) \equiv \int d\tau dx e^{i\tilde{\omega}\tau - ikx} \psi(x, \tau) \quad (20)$$

$$\begin{aligned} \psi(x, \tau) &\equiv \int \frac{dk}{2\pi} \frac{d\tilde{\omega}}{2\pi} e^{-i\tilde{\omega}\tau + ikx} \psi(k, \tilde{\omega}) \\ &= \int_{k, \tilde{\omega}} e^{-i\tilde{\omega}\tau + ikx} \psi(k, \tilde{\omega}) \end{aligned} \quad (21)$$

When going to real time  $t = -i\tau$ , the analytic continuation on frequencies is performed as  $i\tilde{\omega} \rightarrow \omega + i0^+$ .

Performing the Gaussian integration on  $\bar{\psi}_B$  and  $\psi_B$  in equation (16) leads to:

$$Z = Z_B^0 Z_F^{eff} = Z_B^0 \int \mathcal{D}(\bar{\psi}_\sigma, \psi_\sigma) e^{-S_F^{eff}} \quad (22)$$

where the effective action for fermions is

$$\begin{aligned} S_F^{eff} &= \int_{k, \tilde{\omega}} \sum_{\sigma} \bar{\psi}_\sigma(k, \tilde{\omega}) \left[ -i\tilde{\omega} + \frac{k^2}{2m} - \mu \right] \psi_\sigma(k, \tilde{\omega}) \\ &+ g^2 \int_{k, \tilde{\omega}} \int_{k', \tilde{\omega}'} \int_{K, \tilde{\Omega}} \bar{\psi}_\uparrow(k, \tilde{\omega}) \bar{\psi}_\downarrow(K - k, \tilde{\Omega} - \tilde{\omega}) \\ &\times \psi_\downarrow(k', \tilde{\omega}') \psi_\uparrow(K - k', \tilde{\Omega} - \tilde{\omega}') \\ &\times \left[ i\tilde{\Omega} - \frac{K^2}{4m} + 2\mu - \nu \right]^{-1} \end{aligned} \quad (23)$$

and the grand potential for an ideal Bose gas of bare dimers is:

$$\begin{aligned} F_B^0 &\equiv -T \ln Z_B^0 \\ &= TL \int \frac{dK}{2\pi} \ln \left[ 1 - e^{-\beta(K^2/4m - 2\mu + \nu)} \right]. \end{aligned} \quad (24)$$

Due to the fact that only quadratic terms in  $\psi_B$  appear in the original model, the previous result is exact. The resulting effective interaction between the atoms, however, is non-local both in space and time. If we restrict ourselves to the case  $\nu > |\epsilon_\star|$ , together with the broad resonance requirement  $|\epsilon_\star| \gg \epsilon_F$ , we can simplify the effective action  $S_F^{eff}$  to one which is local. Indeed, before resonance,  $2|\mu| \simeq |\epsilon_b| < |\epsilon_\star|$  (see Appendix A) and as an order of magnitude,  $|i\tilde{\Omega}| \sim |K^2/4m| \sim \epsilon_F$ . Therefore, the detuning dominates the denominator of the molecular propagator  $i\tilde{\Omega} - \frac{K^2}{4m} + 2\mu - \nu \simeq -\nu$ , and the effective interaction between fermions becomes

$$-\frac{g^2}{\nu} \int_{k_1, \tilde{\omega}_1} \int_{k_2, \tilde{\omega}_2} \int_{k_3, \tilde{\omega}_3} \bar{\psi}_\uparrow(1 + 2 - 3) \bar{\psi}_\downarrow(3) \psi_\downarrow(2) \psi_\uparrow(1) \quad (25)$$

where (1) is a short notation for  $(k_1, \tilde{\omega}_1)$  and similarly for the other arguments. The total number of atoms can be computed from the partition function (22):

$$\langle N \rangle = -T \frac{\partial \ln Z_B^0}{\partial \mu} - T \frac{\partial \ln Z_F^{eff}}{\partial \mu}. \quad (26)$$

The first term is given by the usual expression for an ideal Bose gas

$$\langle N_B^0 \rangle = 2L \int \frac{dK}{2\pi} \left[ e^{\beta(K^2/4m - (2\mu - \nu))} - 1 \right]^{-1} \quad (27)$$

with  $2\mu - \nu < 0$ . When  $T \rightarrow 0$ , the fraction of atoms that are bound into bare dimers is:

$$\begin{aligned} \frac{\langle N_B^0 \rangle}{N} &\simeq \frac{2}{n} e^{-\beta(\nu - 2\mu)} \int \frac{dK}{2\pi} e^{-\beta K^2/4m} \\ &= \frac{2}{n} \sqrt{\frac{m}{\pi\beta}} e^{-\beta(\nu - 2\mu)} \rightarrow 0. \end{aligned} \quad (28)$$

Therefore

$$N = \langle N \rangle \simeq -T \frac{\partial \ln Z_F^{eff}}{\partial \mu} \quad (29)$$

which shows that  $\mu$  is the chemical potential for the gas of atoms only.

In conclusion, before resonance and under the assumptions that the resonance is broad and that  $\nu > |\epsilon_\star|$ , the system is described by a single channel model of fermions with an action

$$\begin{aligned} S_F^{eff} &= \int_0^\beta d\tau \int dx \left( \sum_{\sigma=\uparrow, \downarrow} \bar{\psi}_\sigma \left[ \partial_\tau - \frac{\partial_x^2}{2m} - \mu_F \right] \psi_\sigma \right. \\ &\quad \left. + g_1 \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right) \end{aligned} \quad (30)$$

where  $\mu_F = \mu$  and  $g_1 = -g^2/\nu < 0$ . This is the action corresponding to the Gaudin-Yang model of 1D fermions interacting via an attractive delta potential [8]. The single dimensionless coupling constant is  $\gamma \equiv mg_1/n$ . In order to describe the BCS-BEC crossover, we will use the parameter  $1/\gamma$  (see Appendix B), the BCS limit corresponding to  $1/\gamma \rightarrow -\infty$  or  $\nu \rightarrow +\infty$ . Due to the condition  $\nu > |\epsilon_\star|$ , before resonance, the parameter  $1/\gamma$  is restricted to:

$$-\infty < \frac{1}{\gamma} < -\frac{n|\epsilon_\star|}{mg^2} \sim -nr_\star. \quad (31)$$

In the broad resonance limit,  $nr_\star \rightarrow 0$  implying that apart from a vanishingly small region close to resonance ( $\nu = 0$  or  $1/\gamma = 0$ ), the Boson-Fermion resonance model, before resonance, is equivalent to the single channel attractive Gaudin-Yang model.

## B. After resonance: integrating out the atoms

After resonance ( $\nu < 0$ ), it is possible to integrate out the fermionic fields and to describe the system in terms of an effective action for dimers only. Formally, this is equivalent to the standard technique used to study the single channel model in 2D or 3D (see e.g. [21, 22, 23, 24]), where via a Hubbard-Stratonovich transformation, it is possible to write the fermionic action in terms of a Bose-field only, which is eventually identified with the order parameter of the superconducting phase. However, it is important to emphasize that the resulting bosonic field, in that context is different from the field  $\psi_B$  appearing in the BFRM defined by the action (17).

Performing the Gaussian integral over fermionic fields [18], one obtains

$$Z = Z_F^0 Z_B^{eff} = Z_F^0 \int \mathcal{D}(\bar{\psi}_B, \psi_B) e^{-S_B^{eff}} \quad (32)$$

where the effective action for bosons is

$$\begin{aligned} S_B^{eff} &= \int_{k, \tilde{\omega}} \bar{\psi}_B(k, \tilde{\omega}) \left[ -i\tilde{\omega} + \frac{k^2}{4m} - 2\mu + \nu \right] \psi_B(k, \tilde{\omega}) \\ &\quad - \ln \det(G_0 G^{-1}) \end{aligned} \quad (33)$$

and  $Z_F^0$  is the partition function for free fermions. We also defined the propagator for non interacting fermions, which in the Nambu representation reads

$$G_0(k, \tilde{\omega}; k', \tilde{\omega}') = \frac{(2\pi)^2 \delta(k - k') \delta(\tilde{\omega} - \tilde{\omega}')}{-i\tilde{\omega} + k^2/2m - \mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (34)$$

and the full propagator  $G$  is related to  $G_0$  by

$$G^{-1} \equiv G_0^{-1} + \Delta \quad (35)$$

where  $\Delta$  is given by:

$$\begin{aligned} \Delta(k, \tilde{\omega}; k', \tilde{\omega}') &\equiv \\ &= g \begin{pmatrix} 0 & \psi_B(k + k', \tilde{\omega} + \tilde{\omega}') \\ \bar{\psi}_B(k + k', \tilde{\omega} + \tilde{\omega}') & 0 \end{pmatrix} \end{aligned} \quad (36)$$

The last term in equation (33) can be expanded into a sum:

$$-\ln \det(G_0 G^{-1}) = \sum_{l=1}^{\infty} \frac{\text{Tr}[(G_0 \Delta)^{2l}]}{2l}. \quad (37)$$

Below, we will evaluate explicitly the  $l = 1$  and  $l = 2$  terms in the sum and, following [21, 23], we will give an order of magnitude estimate for the higher order terms showing that they are negligible in the broad resonance limit.

**$l = 1$  term:**

$$\frac{1}{2} \text{Tr}[(G_0 \Delta)^2] = \int_{k, \tilde{\omega}} \bar{\psi}_B(k, \tilde{\omega}) \psi_B(k, \tilde{\omega}) \Pi(k, \tilde{\omega}) \quad (38)$$

After resonance ( $\nu < 0$ ), we have  $2|\mu| \simeq |\epsilon_b| > |\epsilon_\star| \gg \epsilon_F$ , as discussed in Appendix A. As an order of magnitude

$|i\tilde{\omega}|$  gives a contribution of the order of the kinetic energy  $\sim |k^2/4m| \sim \epsilon_F$ . As a result, we obtain

$$\Pi(k, \tilde{\omega}) \simeq \Pi(0, 0) \simeq \epsilon_b - \nu \quad (39)$$

where we used  $\mu \simeq \epsilon_b/2$  and the bound state equation (7). Thus the  $l = 1$  term just gives rise to an effective chemical potential for the dimers equal to  $2\mu - \epsilon_b$ .

**$l = 2$  term:**

$$\begin{aligned} \frac{1}{4} \text{Tr} [(G_0 \Delta)^4] &= \frac{1}{2} \int_{k_1, \tilde{\omega}_1} \int_{k_2, \tilde{\omega}_2} \int_{k_3, \tilde{\omega}_3} \bar{\psi}_B(1+2-3) \\ &\times \bar{\psi}_B(3) \psi_B(2) \psi_B(1) g_B(1, 2, 3) \end{aligned} \quad (40)$$

in an obvious short hand notation for the associated wave vectors and frequencies, and

$$\begin{aligned} g_B(1, 2, 3) &\equiv g^4 \int_{k, \tilde{\omega}} \left[ (i\tilde{\omega} - \xi_k) \right. \\ &\times (i\tilde{\omega}_2 - i\tilde{\omega} - \xi_{k_2-k}) \\ &\times (i\tilde{\omega}_1 + i\tilde{\omega}_2 - i\tilde{\omega}_3 - i\tilde{\omega} - \xi_{k_1+k_2-k_3-k}) \\ &\left. \times (-i\tilde{\omega}_2 + i\tilde{\omega}_3 + i\tilde{\omega} - \xi_{-k_2+k_3+k}) \right]^{-1} \end{aligned} \quad (41)$$

with  $\xi_k \equiv k^2/2m - \mu$ . Using again the condition characterizing a broad resonance, we see that the momentum and frequency dependance of the interaction is irrelevant. This implies that

$$\begin{aligned} g_B(1, 2, 3) &\simeq g_B(k_j = 0, \tilde{\omega}_j = 0; j = 1, 2, 3) \\ &\simeq g^4 \int_{k, \tilde{\omega}} [\tilde{\omega}^2 + \xi_k^2]^{-2} = \frac{3g^4 \sqrt{m}}{8|\epsilon_b|^{5/2}} \end{aligned} \quad (42)$$

with  $\mu \simeq \epsilon_b/2$ .

**$l \geq 3$  term:** For all  $l$ , we obtain the following estimate for the corresponding term in the sum (37):

$$t_l \equiv \frac{1}{2l} \text{Tr} [(G_0 \Delta)^{2l}] \sim \frac{g^{2l}}{|\epsilon_b|^{2l-1} r_b} \frac{k_F^{l-1}}{\epsilon_F} \sim (nr_b)^{l-3}. \quad (43)$$

For  $l = 1$  and  $l = 2$ , in the broad resonance limit where  $1 \gg nr_\star > nr_b$ , we obtain that  $t_1$  and  $t_2 \gg 1$ . For  $l \geq 3$ , the ratio  $t_2/t_l \gg 1$  in the broad resonance limit and the corresponding terms can therefore be neglected.

In the broad resonance limit, the effective action for bosons becomes

$$\begin{aligned} S_B^{eff} &= \int_0^\beta d\tau \int dx \left( \bar{\psi}_B \left[ \partial_\tau - \frac{\partial_x^2}{4m} - (2\mu - \epsilon_b) \right] \psi_B \right. \\ &\left. + \frac{g_B}{2} \bar{\psi}_B \bar{\psi}_B \psi_B \psi_B \right) \end{aligned} \quad (44)$$

where  $g_B \equiv 3g^4 \sqrt{m}/8|\epsilon_b|^{5/2}$  describes a repulsive interaction between the strongly bound dimers. From the partition function (32) and the preceding effective action, we can obtain the average total number of atoms:

$$\langle N \rangle = -T \frac{\partial \ln Z_F^0}{\partial \mu} - T \frac{\partial \ln Z_B^{eff}}{\partial \mu}. \quad (45)$$

The first term is given by the usual expression for an ideal Fermi gas:

$$\langle N_F^0 \rangle = L \int \frac{dk}{2\pi} \left[ e^{\beta(k^2/2m - \mu)} + 1 \right]^{-1}. \quad (46)$$

In the limit  $T \rightarrow 0$  and using the fact that after resonance  $\mu \simeq \epsilon_b/2$ , the fraction of atoms that are unbound is exponentially small. Therefore, (45) becomes

$$\frac{N}{2} = \frac{\langle N \rangle}{2} \simeq -T \frac{\partial \ln Z_B^{eff}}{\partial 2\mu} \quad (47)$$

which shows that  $2\mu$  is the chemical potential for the gas of dimers only. We now shift the zero of energy of the many-body system by an amount  $-N\epsilon_b/2$ , and accordingly define  $\mu_B \equiv 2\mu - \epsilon_b$  as the new chemical potential for dimers.

In conclusion, after resonance and under the assumption that the resonance is broad, the system is described by a single channel model of bosons (i.e. dimers) with an action

$$\begin{aligned} S_B^{eff} &= \int_0^\beta d\tau \int dx \left( \bar{\psi}_B \left[ \partial_\tau - \frac{\partial_x^2}{2m_B} - \mu_B \right] \psi_B \right. \\ &\left. + \frac{g_B}{2} \bar{\psi}_B \bar{\psi}_B \psi_B \psi_B \right) \end{aligned} \quad (48)$$

where  $m_B \equiv 2m$ ,  $\mu_B = 2\mu - \epsilon_b$  and  $g_B = 3g^4 \sqrt{m}/8|\epsilon_b|^{5/2}$  [25]. This is the action corresponding to the Lieb-Liniger model of  $N_B \equiv N/2$  bosons of mass  $m_B$  interacting via a repulsive delta potential [9]. The single dimensionless coupling constant is  $\gamma \equiv mg_B/n$  and the BEC limit corresponds to  $1/\gamma \rightarrow +\infty$  or  $\nu \rightarrow -\infty$ . Because  $g_B \sim 1/mr_\star$  when  $\nu \rightarrow 0^-$  (see Appendix B), the parameter  $1/\gamma$  is restricted to:

$$nr_\star < \frac{1}{\gamma} < +\infty \quad (49)$$

In the broad resonance limit,  $nr_\star \rightarrow 0$  implying that apart from a vanishingly small region close to resonance the Boson-Fermion resonance model, after resonance, is equivalent to the single channel repulsive Lieb-Liniger model for dimers.

## V. DISCUSSION

It was recently shown [6, 7] that interacting fermions in a quasi-1D geometry and in presence of a Feshbach resonance map onto the modified Gaudin-Yang model in the limit of very strong confinement  $na_\perp \ll 1$ . In the present paper, we have seen that the Boson-Fermion resonance model in 1D is also described by the same model in the limit of a broad resonance  $nr_\star \ll 1$ . Close to resonance, the system behaves as a Tonks-Girardeau gas (or impenetrable Bose gas) [27] of dimers [6, 7, 28]. Around resonance, there is a vanishingly small region

$1/\gamma \in [-nr_*, nr_*]$ , which is not described by the modified Gaudin-Yang model. The relation between the parameters  $g_1$  and  $g_B$  of the modified Gaudin-Yang model and those of the original system – either the quasi-1D single channel model or the 1D Boson-Fermion resonance model – is different. Deep in the BEC limit, the two-body bound state of the BFRM is given by  $\epsilon_b \simeq \nu$  and is populated by pairs of fermions. All fermions are bound into dimers and the scattering properties of dimers have no direct relation to the scattering properties of fermions, in particular  $g_B$  is not simply proportional to  $g_1$ . This is in contrast to the 3D single channel model, where it is known that the dimer-dimer scattering length is proportional to the fermion-fermion scattering length (see, e.g., [29]). The equivalent result for the quasi-1D single channel is still under investigation [30]. Nevertheless, in the BEC limit, one expects that  $g_B \approx 0.6 g_1$  [6, 7].

The above scenario for a BCS-BEC crossover can be realized, e.g., in an experiment with ultra-cold gases confined in a quasi-1D trap either by tuning the 3D  $s$ -wave scattering length via a magnetic field in order to cross the CI resonance, as discussed in [6], or by photoassociation [31], which corresponds to a direct implementation of the BFRM. As mentioned before, a description of the resulting 1D BCS-BEC crossover by the modified Gaudin-Yang model is possible under the condition (19) for a broad resonance. Specifically, the realization using a CI resonance in a tight waveguide requires a sufficiently dilute gas with  $na_\perp \ll 1$ . Taking typical values of order 50 nm for the transverse oscillator length which have been realized very recently in bosonic 1D gases [4, 33], this requires densities in the range of much less than 20 atoms per micron. In the case of photo-association, i.e. an optically induced resonance, the requirement is, that the effective 1D Rabi frequency  $g\sqrt{n}$  is much larger than the Fermi energy. Using estimates for the Rabi-frequency taken over from photassociation of  $^{87}\text{Rb}$  in 3D [34], a rough estimate shows that the condition of a broad resonance can also be reached here. In particular the fact that the Franck-Condon overlap is enhanced in a 1D situation helps realizing this limit.

We acknowledge useful discussions with Walter Rantner, Stefano Cerrito and Andrea Micheli. Laboratoire de Physique des Solides is a mixed research unit (UMR 8502) of the CNRS and the Université Paris-Sud in Or-

say.

## Appendix A

The estimates of  $\mu$  used in the present article come from identifying  $\delta\mu \equiv \mu - \epsilon_b/2$  (when in the broad resonance limit) with the chemical potential in the modified Gaudin-Yang model [6]. The chemical potential obtained from [6] gives the following estimate for  $\delta\mu$ :

$$\delta\mu/\epsilon_F \simeq \begin{cases} 1 & \text{when } 1/\gamma \rightarrow -\infty \text{ BCS limit} \\ 1/4 & \text{when } 1/\gamma \rightarrow 0 \text{ on resonance} \\ \gamma/4\pi^2 & \text{when } 1/\gamma \rightarrow +\infty \text{ BEC limit} \end{cases}$$

## Appendix B

In this appendix, we discuss the behavior of  $1/\gamma$  as a function of  $\nu$ . Before resonance  $\gamma = mg_1/n = -mg^2/n\nu$ , which implies:

$$\frac{1}{\gamma} = -\frac{nr_*}{2^{3/2}} \frac{\nu}{|\epsilon_*|}. \quad (50)$$

In the BCS limit  $\nu \rightarrow +\infty$ ,  $1/\gamma \rightarrow -\infty$  and on resonance  $\nu \rightarrow 0^+$ ,  $1/\gamma \rightarrow 0^-$ . We assumed that  $|\nu| > |\epsilon_*|$ , which implies  $1/|\gamma| > nr_*/2^{3/2}$  with  $nr_* \ll 1$  (broad resonance limit), so that indeed, close to resonance  $1/\gamma \simeq 0$ .

After resonance,  $\gamma = mg_B/n$  and equation (42) can be rewritten [32]

$$g_B = \frac{3}{\sqrt{2}mr_*} \left( \frac{\epsilon_b}{\epsilon_*} \right)^{-5/2} \quad (51)$$

which implies:

$$\frac{1}{\gamma} = \frac{\sqrt{2}nr_*}{3} \left( \frac{\epsilon_b}{\epsilon_*} \right)^{5/2}. \quad (52)$$

Close to resonance  $\nu \rightarrow 0^-$ ,  $g_B \simeq 3/\sqrt{2}mr_*$  and  $1/\gamma \simeq \sqrt{2}nr_*/3$  with  $nr_* \ll 1$  (broad resonance limit), so that again,  $1/\gamma \simeq 0$  close to resonance. In the BEC limit  $\nu \rightarrow -\infty$ ,  $g_B \simeq 3\epsilon_*^{5/2}/\sqrt{2}mr_*\nu^{5/2}$  and  $1/\gamma \simeq \sqrt{2}nr_*\nu^{5/2}/3\epsilon_b^{5/2} \rightarrow +\infty$ .

- 
- [1] C.A. Regal, M. Greiner, and D.S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).  
[2] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 120401 (2004); M.W. Zwierlein, C.A. Stan, C.H. Schunck, S.M.F. Raupach, A.J. Kerman, and W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004); J. Kinast, S.L. Hemmer, M.E. Gehm, A. Turlapov, J.E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004); T. Bourdel, L. Khaykovich,

- J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S.J.J.M.F. Kokkelmans, and C. Salomon, Phys. Rev. Lett. **93**, 050401 (2004).  
[3] D.M. Eagles, Phys. Rev. **186**, 456 (1969); A.J. Leggett, J. Phys. C (Paris) **41**, 7 (1980); P. Nozières, S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985); M. Randeria, in *Bose-Einstein Condensation*, ed. by A. Griffin, D.W. Snoke, S. Stringari (Cambridge University Press, Cambridge, 1995) and references therein.

- [4] H. Moritz, T. Stöferle, M. Köhl, and T. Esslinger, Phys. Rev. Lett. **91**, 250402 (2003).
- [5] J. Reichel, and J.H. Thywissen, to appear in J. Phys. IV France **116**, xxxx (2004); arXiv:cond-mat/0310330.
- [6] J.N. Fuchs, A. Recati, and W. Zwerger, Phys. Rev. Lett. **93**, 090408 (2004).
- [7] I.V. Tokatly, Phys. Rev. Lett. **93**, 090405 (2004).
- [8] M. Gaudin, Phys. Lett. **24A**, 55 (1967); C.N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); see also M. Gaudin, *La fonction d'onde de Bethe* (Masson, Collection CEA, Paris, 1983).
- [9] E.H. Lieb, and W. Liniger, Phys. Rev. **130**, 1605 (1963); E.H. Lieb, Phys. Rev. **130**, 1616 (1963).
- [10] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
- [11] R. Friedberg, and T.D. Lee, Phys. Rev. B **40**, 6745 (1989).
- [12] J. Ranninger, J. M. Robin and M. Eschrig, Phys. Rev. Lett. **74**, 4027 (1995), and references therein.
- [13] M. Holland, S.J.J.M.F. Kokkelmans, M.L. Chiofalo, and R. Walser, Phys. Rev. Lett. **87**, 120406 (2001); E. Timmermans, K. Furuya, P.W. Milonni, and A.K. Kerman, Phys. Lett. A **285**, 228 (2001).
- [14] In the 3D version of the model defined by the Hamiltonian (1), a renormalized detuning appears in the bound state energy equation instead of the bare one as in the 1D case. This is due to the fact that a delta interaction is not a valid 3D potential and needs to be renormalized [13].
- [15] R.A. Duine, and H.T.C. Stoof, J. Opt. B: Quantum Semiclass. Opt. **5**, S212 (2003).
- [16] T. Bergeman, M.G. Moore, and M. Olshanii, Phys. Rev. Lett. **91**, 163201 (2003).
- [17]  $\zeta(1/2, x)$  is a particular Hurvitz zeta function and  $\zeta(1/2, 1) \simeq -1.4603$ , see [16]. Note the  $\sqrt{2}$  difference in the definition of  $a_{\perp}$  with [10, 16], which accounts for  $A \equiv -\zeta(1/2, 1)/\sqrt{2} \simeq 1.0326$ .
- [18] See, for example, A. Altland, and B. Simons, *Concepts of Theoretical Solid State Physics*, <http://www.thp.uni-koeln.de/~alexal/teaching.html> (2001) or J.W. Negele, and H. Orland, *Quantum Many-Particle Systems* (Addison Wesley, Redwood City, 1988).
- [19] G.M. Bruun, arXiv:cond-mat/0401497 (2004); S. De Palo, M.L. Chiofalo, M.J. Holland, and S.J.J.M.F. Kokkelmans, arXiv:cond-mat/0404672 (2004); R.B. Diener, and T.L. Ho, arXiv:cond-mat/0405174 (2004); S. Simonucci, P. Pieri, and G.C. Strinati, arXiv:cond-mat/0407600 (2004).
- [20] Pistolesi and Strinati call it “the bosonization condition” [21]. In 3D, this condition corresponds to requiring that the effective range of the interaction between atoms mediated by molecules is small, i.e.  $n^{1/3}|r_{eff}| \ll 1$  with  $r_{eff} = -8\pi/m^2g^2$ , see Bruun [19] and S. De Palo et al. [19].
- [21] F. Pistolesi, and G.C. Strinati, Phys. Rev. B **53**, 15168 (1996).
- [22] M. Drechsler, and W. Zwerger, Ann. Phys. (Germany) **1**, 15 (1992).
- [23] A.A. Gorbatshevich, and I.V. Tokatly, Sov. Phys. JETP **76**, 347 (1993).
- [24] C.A.R. Sá de Melo, M. Randeria and J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
- [25] In the 3D resonance model,  $g_B$  is given by the same expression as in (42) except that the  $k$  integral is over three dimensions. This gives  $g_B = m^{3/2}g^4/16\pi|\nu|^{3/2}$  in the BEC limit, where  $\epsilon_b \simeq \nu$ . Using the previous result, to calculate the Bogoliubov sound velocity  $c \equiv \sqrt{g_B n_B/m_B}$  with  $n_B \equiv n/2$ , we recover the result of Ohashi and Griffin for the sound velocity in the BEC limit of the 3D resonance model, see equation (4.20) in [26].
- [26] Y. Ohashi, and A. Griffin, Phys. Rev. A **67**, 063612 (2003).
- [27] M. Girardeau, J. Math. Phys. (N.Y.) **1**, 516 (1960).
- [28] G.E. Astrakharchik, D. Blume, S. Giorgini, and L.P. Pitaevskii, Phys. Rev. Lett. **93**, 050402 (2004).
- [29] D.S. Petrov, C. Salomon, G.V. Shlyapnikov, Phys. Rev. Lett. **93**, 090404 (2004).
- [30] C. Mora, R. Egger, A.O. Gogolin, and A. Komnik, arXiv:cond-mat/0406627 (2004); C. Mora, private communication.
- [31] M. Mackie, E. Timmermans, R. Côté, and J. Javanainen, Optics Express **8**, 118 (2001) and references therein.
- [32] We note that this is of the form  $(ma_{\perp})^{-1}f(|\epsilon_b|/2\omega_{\perp})$ , with the identification  $r_{\star} \leftrightarrow a_{\perp}$  and  $|\epsilon_{\star}| \leftrightarrow 2\omega_{\perp}$ , as discussed in [7].
- [33] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch and I. Bloch, Nature **429**, 277 (2004); Toshiya Kinoshita, Trevor Wenger, and David S. Weiss, Science **305**, 1125 (2004).
- [34] G. Thalhammer, M. Theis, K. Winkler, R. Grimm, J. H. Denschlag, available at arXiv.org/ cond-mat/0409552.